MISG 2021 Progress Report

Masks and the spread of droplets and airborne virons.

D. P. Mason, N. Fowkes, T. Myers, N. Hale, I. Griffiths, M. Khalique, N. Modhien, H. Zha, E. Mubai, K. Born, T. Magodi, H. Bhana, F. Rakotoniaina, P. Chiwira

February 6, 2021

Table of Contents

Table of Contents

Problem Statement

Dr. Fauci recently advised that, "If you have a physical covering with one layer, you put another layer on, it just makes common sense that it likely would be more effective. That's the reason why you see people either double masking or doing a version of an N-95."

To test this, we wish to construct a model of two masks (filters).

Table of Contents

Mathematical Models

Rigid Filtration Problem

Droplet Trajectories

Elastic Mask Problem

Mask Design Problem

Table of Contents

Rigid Filtration Model

Properties of the Model

- Flow with adsorption in one-dimension
- Masks are rigid
- Assume masks are in perfect contact

Advection-Diffusion system with mass sink

$$\frac{\partial c}{\partial t} - \frac{\partial}{\partial x}\left[D\frac{\partial^2 c}{\partial x^2} - (uc) \right] = -\gamma(q^* - q), \tag{1}$$

where c is the average concentration of water droplets in the porous media, q is the amount adsorbed onto the filter, q^* is the saturation value, γ is the adsorption rate, and D is the diffusion coefficient. D depends on the porosity, permeability, air speed, and droplet size.

Steady-state

$$\frac{\partial}{\partial x}\left[D\frac{\partial c}{\partial x}-(uc)\right]=\gamma(q^{*}-q) \tag{2}$$

The Langmuir isotherm

$$q=\frac{kc}{1+kc}. \tag{3}$$

$kc \ll 1$ permits a linear, constant coefficients governing equation.

At the inlet there is continuity of flux

$$uc|_{x=0^-} = \left(uc - D\frac{\partial c}{\partial x} \right)\bigg|_{x=0^+} . \tag{4}$$

At the outlet $x = L$

$$\left.\frac{\partial c}{\partial x}\right|_{x=L} = 0. \tag{5}$$

In the case of two masks in perfect contact, at the interface (call this L_1 and outlet at L_2) we apply two governing equations with different values for D, γ, c^*

At the interface we impose continuity of concentration and flux

$$[c_i]_{x=L_1} = 0, \qquad \left[uc_i - D_i\frac{\partial c_i}{\partial x} \right]_{x=L_1} = 0. \tag{6}$$

Velocity

Darcy's law

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x} = -\frac{k}{\mu}\frac{\Delta p}{L} \tag{7}$$

k is the permeability of the mask, μ the dynamic viscosity of the air and Δp the pressure drop across the mask.
Two layers – L_1, L_2, permeability k_1, k_2, driven by a pressure drop $\Delta p = p_2 - p_0 < 0$
At the interface we denote the unknown pressure as p_1. Mass conservation indicates

$$u = -\frac{k_1}{\mu}\left(\frac{p_1 - p_0}{L_1}\right) = -\frac{k_2}{\mu}\left(\frac{p_2 - p_1}{L_2}\right) \tag{8}$$

Hence ...

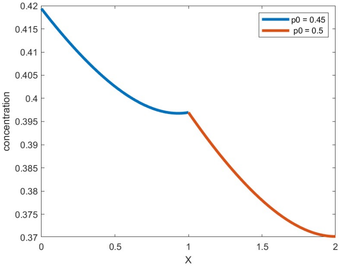
Ease of breathing

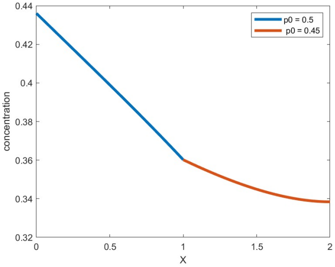
$$u = -\frac{k_1 k_2}{\mu}\left(\frac{\Delta p}{k_1 L_2 + k_2 L_1}\right) \tag{9}$$

In terms of ease of breathing it doesn't matter where the layer is k, L are interchangeable.

But the layer position does affect the outlet concentration ...

Which layer first?





Left: 2nd layer most porous   Right: 1st layer most porous

Table of Contents

Where does my spit go?

Consider mouth some distance from an impermeable plate ...
Steady-state Navier–Stokes equations

$$Re\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{10}$$

$$Re\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}. \tag{11}$$

Similarity variable $\eta=\sqrt{Re}y$ and stream function

$$\psi=\frac{x}{\sqrt{Re}}f(\eta) \tag{12}$$

such that $u=\psi_y,\ v=-\psi_x$.

Missing spit!

$$xRe\left[f_\eta^2 - ff_{\eta\eta}\right] = -\frac{\partial p}{\partial x} + xRef_{\eta\eta\eta} \tag{13}$$

$$Re\left[\frac{1}{\sqrt{Re}}ff_\eta\right] = -\sqrt{Re}p_\eta - \sqrt{Re}f_{\eta\eta} \tag{14}$$

To remove the x dependence in the first equation

$$p = \pm Re\frac{x^2}{2} + g(\eta). \tag{15}$$

The second equation integrates immediately

$$\frac{f^2}{2} = -p - f_\eta + h(x) \tag{16}$$

Comparison of the two expressions for p leads to

$$p - p_0 = -\left[Re\frac{x^2}{2} + f_\eta + \frac{f^2}{2}\right] \tag{17}$$

where we have chosen the negative branch for the x^2 term

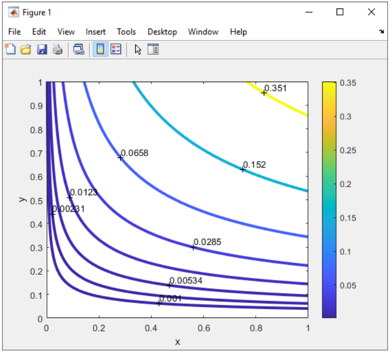
Airflow

ODE for $f(\eta)$

$$f_{\eta\eta\eta} + ff_{\eta\eta} - f_\eta^2 + 1 = 0 \tag{18}$$

This is subject to

$$f(0) = f_\eta(0) = 0, \qquad f_\eta(\infty) = 1 \tag{19}$$



Back to spit

Droplet motion

$$St\,\underline{x}_{tt} = \frac{C_D}{2}(\underline{u}-\underline{x}_t)\left|\underline{u}-\underline{x}_t\right| \tag{20}$$

The drag coefficient

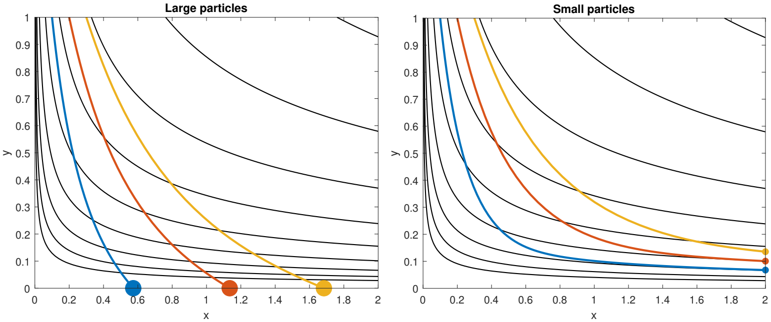
$$C_D = 2\left[1.849Re_p^{-0.31} + 0.293Re_p^{0.06}\right]^{3.45} \tag{21}$$

The Stokes and Reynolds' numbers are

$$St = \frac{4}{3}\frac{\rho_p}{\rho_f}\frac{a}{L} \qquad Re_p = 2\epsilon Re\left|\underline{u}-\underline{x}_t\right| \qquad Re = \frac{\rho_f UL}{\mu} \tag{22}$$

Release different size droplets from $(x_0, 1)$ with velocity $\mathrm{u}_0 = (0, -1)$ and determine whether they hit the mask (i.e. reach $y = 0$) or move to the side for a sufficient distance to escape the mask.

Cool video



Smaller particles are diverted, whereas larger particles are more likely to impact at the mask surface.

Need to include permeability effect - obviously higher permeability implies more droplets entering - similar to moving mask up into streamlines

Recommendations

If we have two masks in intimate contact (or one mask made of two layers), with different properties then
1) It makes no difference for the ease of breathing which is first.
2) It does make a difference to the removal of droplets.
But ... if two masks really need to investigate flow in air gap
Does more leak out of the side than enter the second layer?
(We ran out of time on this, but seems important)

Study of droplet motion indicates higher permeability gets droplets into mask

Put the most permeable part near the face!!!!

MISG South Africa 2021

Table of Contents

Two Elastic Masks

How does having two masks with different material properties affect the flux through the masks?

Design parameters are undeformed permeability, and response of permeability to deformation

Extend the work of Köry et al.

Two Elastic Masks

Two masks

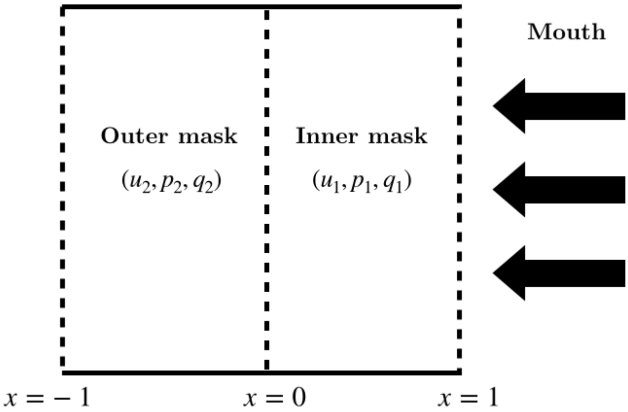
No gap between them

Laterally uniform, i.e. 1D sufficient

Compressible, with small deformation assumed

Permeabilities depend linearly on deformation

Steady, i.e. poroelastic timescale $\ll$ breathing/coughing timescale


Mouth
Outer mask
(u_2, p_2, q_2)
Inner mask
(u_1, p_1, q_1)
$x = -1$
$x = 0$
$x = 1$

Governing equations

linear Navier equation:

$$(\lambda_i + 2\mu_i)\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} = \frac{\mathrm{d}p_i}{\mathrm{d}x}$$

for $i = 1, 2$. η_i =viscosity, λ_i, μ_i =effective elastic coefficients, κ_i =permeability

Governing equations

linear Navier equation:

$$(\lambda_i + 2\mu_i)\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} = \frac{\mathrm{d}p_i}{\mathrm{d}x}$$

Darcy's law:

$$q_i = \frac{\kappa_i}{\eta_i}\frac{\mathrm{d}p_i}{\mathrm{d}x}$$

for $i = 1, 2$. η_i =viscosity, λ_i, μ_i =effective elastic coefficients, κ_i =permeability

Governing equations

linear Navier equation:

$$(\lambda_i + 2\mu_i)\frac{\mathrm{d}^2 u_i}{\mathrm{d}x^2} = \frac{\mathrm{d}p_i}{\mathrm{d}x}$$

Darcy's law:

$$q_i = \frac{\kappa_i}{\eta_i}\frac{\mathrm{d}p_i}{\mathrm{d}x}$$

Continuity equation:

$$\frac{\mathrm{d}q_i}{\mathrm{d}x} = 0$$

for $i = 1, 2$. η_i =viscosity, λ_i, μ_i =effective elastic coefficients, κ_i =permeability

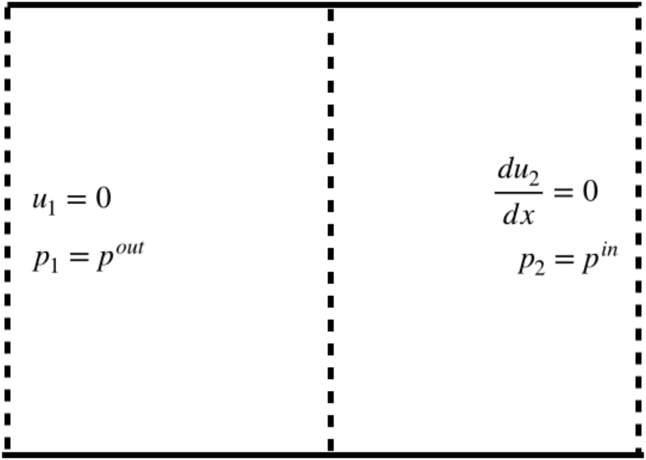
Permeability-deformation constitutive law

undeformed permeability k_i

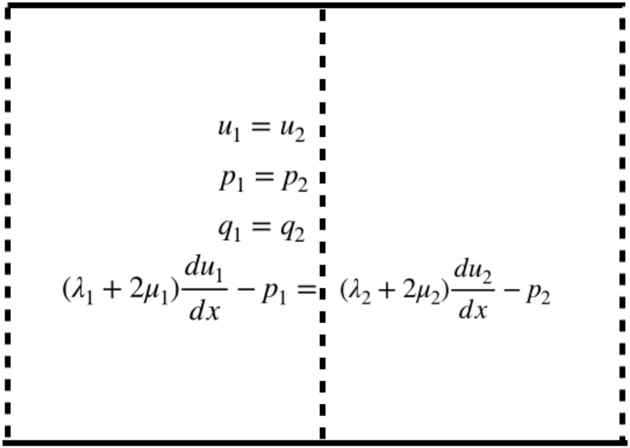
permeability scales linearly with deformation gradient via α_i

$$\kappa_i = k_i\left(1+\alpha_i\frac{\mathrm{d}u_i}{\mathrm{d}x}\right)$$

Boundary conditions



Interface conditions



Solutions

Depend on values of λ_i, μ_i, etc.

$$u_i(x) = \pm \frac{1}{\lambda_i + 2\mu_i} \frac{(2A_i x - 2C_{1i})^{3/2}}{3A_i^2} + C_{3i}x + C_{4i}$$

$$p_i(x) = C_{2i} \pm \frac{(2A_i x - 2C_{1i})^{1/2}}{A_i}$$

Where $A_i = \frac{\kappa_i \alpha_i}{\eta_i q_i(\lambda_i + 2\mu_i)}$ and $C_{ji}, \quad j = 1,2,3,4$ are constants of integration to be determined from boundary and interface conditions. Considering limit of rigid mask, must choose plus sign.