Inverse Laplace Transform Examples

Example 1: Find $\mathcal{L}^{-1}\left\{\frac{1}{z^3(z^2+1)^2}\right\}$.

Let $F(z) = \frac{1}{z^3(z^2+1)^2}$. Then $|F(z)| \to 0$ as $|z| \to \infty$ (because $|F(z)| \simeq \frac{1}{|z^7|}$ for large $|z|$). $F(z)$ has a triple pole at $z = 0$ and double poles at $z = \pm i$.

At $z = 0$:

$$\begin{aligned}
e^{zt}F(z) &= \frac{1}{z^3}e^{zt}\frac{1}{1+2z^2+z^4} \\
&= \frac{1}{z^3}\left\{1 + zt + \frac{(zt)^2}{2} + \cdots\right\}\left\{1 - 2z^2 + \cdots\right\} \\
&= \frac{1}{z^3}\left\{1 + zt + z^2\left(\frac{t^2}{2} - 2\right) + \cdots\right\}.
\end{aligned}$$

Therefore at $z = 0$ the residue is $\frac{t^2}{2} - 2$.

At $z = i$:

Put $w = z - i$ so $z = w + i$ and expand $e^{zt}F(z)$ in powers of w.

$$\begin{aligned}
e^{zt}F(z) &= \frac{e^{(w+i)t}}{(w+i)^3[(w+i)^2+1]^2} \\
&= \frac{e^{wt}e^{it}}{(w+i)^3w^2(w+2i)^2} \\
&= \frac{e^{it}}{w^2}\left\{1 + wt + \frac{(wt)^2}{2} + \cdots\right\}\frac{1}{i^3}\frac{1}{(1+\frac{w}{i})^3}\frac{1}{(2i)^2}\frac{1}{(1+\frac{w}{2i})^2} \\
&= \frac{e^{it}}{w^2}\left\{1 + wt + \frac{(wt)^2}{2} + \cdots\right\}i\left\{1 - \frac{3w}{i} + \cdots\right\}\left(\frac{-1}{4}\right)\left\{1 - \frac{2w}{2i} + \cdots\right\} \\
&= \frac{e^{it}}{w^2}\left(\frac{-i}{4}\right)\left\{1 + w(t + 3i + i) + \cdots\right\} \\
&= \frac{e^{it}}{(z-i)^2}\left(\frac{-i}{4}\right)\left\{1 + (z-i)(t+4i) + \cdots\right\}.
\end{aligned}$$

To expand $\frac{1}{(1+\frac{w}{i})^3}$ and $\frac{1}{(1+\frac{w}{2i})^2}$ above, we used the binomial theorem which says $(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.

Therefore the residue of $e^{zt}F(z)$ at $z = i$ is $\frac{-ie^{it}}{4}(t+4i) = e^{it}\left(1 - \frac{it}{4}\right)$.

Similarly, the residue at $z = -i$ is $e^{-it}\left(1 + \frac{it}{4}\right)$, and so

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{z^3(z^2+1)^2}\right\} &= \frac{t^2}{2} - 2 + e^{it}\left(1 - \frac{it}{4}\right) + e^{-it}\left(1 + \frac{it}{4}\right) \\
&= \frac{t^2}{2} - 2 + \left(e^{it} + e^{-it}\right) - \frac{it}{4}\left(e^{it} - e^{-it}\right) \\
&= \frac{t^2}{2} - 2 + 2\cos t + \frac{t}{2}\sin t.
\end{aligned}$$

Example 2: Find $\mathcal{L}^{-1}\left\{\frac{1}{z(1-e^{-z})}\right\}$. (Similar but not the same as the second class example.)

Let $F(z) = \frac{1}{z(1-e^{-z})}$, then $|F(z)| \to 0$ as $|z| \to \infty$. $F(z)$ has poles whenever $z(1-e^{-z}) = 0$, which is when $z = 0$ and when $z = 2in\pi$, $n \in \mathbb{Z}$. These are all simple poles except when $n = 0$, which is a double pole.

At $z = 0$:

$$\begin{aligned}
\frac{e^{zt}}{z(1-e^{-z})} &= \frac{1}{z}\left\{1 + zt + \frac{(zt)^2}{2} + \cdots\right\}\frac{1}{1-\left(1 - z + \frac{z^2}{2} - \cdots\right)} \\
&= \frac{1}{z}\left\{1 + zt + \frac{(zt)^2}{2} + \cdots\right\}\frac{1}{z - \frac{z^2}{2} + \cdots} \\
&= \frac{1}{z^2}\left\{1 + zt + \frac{(zt)^2}{2} + \cdots\right\}\left\{1 + \frac{z}{2} + \cdots\right\} \\
&= \frac{1}{z^2}\left\{1 + z\left(t + \frac{1}{2}\right) + \cdots\right\},
\end{aligned}$$

so the residue at $z = 0$ is $t + \frac{1}{2}$.

At $z = 2in\pi$, $n \in \mathbb{Z}$ and $n \neq 0$, the residue is:

$$\begin{aligned}
\lim_{z\to 2in\pi}(z - 2in\pi)\frac{e^{zt}}{z(1-e^{-z})} &= \lim_{z\to 2in\pi}\frac{e^{zt}}{z}\frac{(z-2in\pi)}{(1-e^{-z})} \\
&= \frac{e^{2in}}{2in\pi}\lim_{z\to 2in\pi}\frac{(z-2in\pi)}{(1-e^{-z})} \text{ which is of the form } \frac{0}{0} \\
&= \frac{e^{2in}}{2in\pi}\lim_{z\to 2in\pi}\frac{1}{e^{-z}} \text{ by L'Hopital} \\
&= \frac{e^{2in}}{2in\pi} \text{ since } e^{-2in\pi} = 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{z(1-e^{-z})}\right\} &= t + \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2in}}{2in\pi} \\
&= t + \frac{1}{2} + \sum_{n=-\infty}^{1}\frac{e^{2in}}{2in\pi} + \sum_{n=1}^{\infty}\frac{e^{2in}}{2in\pi} \\
&= t + \frac{1}{2} + \sum_{n=1}^{\infty}\frac{e^{-2in}}{-2in\pi} + \sum_{n=1}^{\infty}\frac{e^{2in}}{2in\pi} \\
&= t + \frac{1}{2} + \sum_{n=1}^{\infty}\frac{e^{2in} - e^{-2in}}{2in\pi} \\
&= t + \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\pi t)}{n}.
\end{aligned}$$

Done.